Playing Konane Mathematically: A Combinatorial Game-Theoretic Analysis

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August 7, 1995

Technical Report
MSR-TR-95-24

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Introduction

This article presents a combinatorial game-theoretic analysis of Konane, an ancient Hawaiian stone-jumping game. Combinatorial game theory [Berlekamp et al. 1982] applies particularly well to Konane because the first player unable to move loses and because a game often can be divided into independent subgames whose outcomes can be combined to determine the outcome of the entire game. By contrast, most popular modern games violate the assumptions of combinatorial game-theoretic analysis. This article describes the game of Konane and the ideas of combinatorial game theory, derives values for a number of interesting positions, shows how to determine when a game can be divided into noninteracting subgames, and provides anthropological details about Konane.

The Game of Konane

Konane is played with black and white game pieces on a rectangular grid of locations. Initially, all the locations are occupied, with black and white pieces alternating. The game begins with the removal of two adjacent pieces (one black and one white). Thereafter, players alternate moves, one player moving the black pieces and the other player moving the white. A player moves by jumping a piece over an adjacent opposing piece into an empty location just beyond. The jumped piece is removed from the board. Multiple jumps are permitted within a single move so long as the jumped pieces lie in a straight line, each separated by exactly one empty location. Jumps are always made along a single rank or file, never diagonally and never in multiple directions in a move. The first player unable to move loses. (The number of pieces jumped over is irrelevant to scoring.) Thus, the goal of the game is jump opposing pieces while not affording the opponent the opportunity to jump one's own pieces.

The Appendix gives information on board size, how to remove the initial two pieces, and anthropological details regarding Konane. A simpler variant, which we call Modern Konane (contrasted to Ancient Konane), prohibits multiple jumps in a single move: Each move consists of jumping exactly one adjacent enemy piece. Modern Konane is the game that Sizer [1991, 58] calls Konane; an earlier version of this article [Ernst and Berlekamp 1993] incorrectly followed Sizer's terminology. When a statement applies equally well to both variants of the game, we use "Konane" without a modifier. Except where noted, our analysis assumes an unbounded board, so we can ignore the constraints imposed by its edges.

Combinatorial Game Theory

This section briefly introduces combinatorial game theory. For a more detailed exposition, see one of the more complete treatments [Berlekamp et al. 1982; Conway 1976; Guy 1991a; Knuth 1974].

Given a position in a game, we would like to know which player wins if play starts from that position. Given a particular arrangement of pieces on a game board, which player has the advantage? Combinatorial game theory answers this question for two-player games in which both players have complete information, chance plays no part, players move alternately, the first player unable to move loses, and the game is guaranteed to end. (Guy [1991b] gives a slightly more exhaustive list of conditions.)

Konane fits these conditions ideally. Modern games susceptible to combinatorial game-theoretic analysis include Go [Berlekamp and Wolfe 1994], Domineering [Berlekamp 1988a], and Dots-and-Boxes [Berlekamp 1988b; Nowakowski 1990].

Combinatorial game theory assigns to each game position a value indicating which player wins. The value, frequently a number, indicates roughly how many moves ahead the winning player is—that is, how many "free" moves the winner could take after the loser is no longer able to move. By convention, an advantage for Black is represented as a positive value.

We call the two players Black and White, after the colors of their pieces, and assume that they play perfectly. Given a particular game position (or game), there are four possibilities for who wins:

- Black: *Positive* games are won by Black regardless of who moves first. For instance, when the game ○●○ is played on an unbounded board, White is unable to move, but Black can make one move (by jumping over either white piece); the game's value is 1.

- White: White always wins *negative* games such as ●○●○●, which has value -2: White can move twice, while Black is helpless.

- the second player to move: In a *zero* game, the second player to move can always win. The simplest game of all, in which no moves are possible and the player obliged to move loses, has value 0.

- the first player to move: These are the *fuzzy* games, and they have nonnumeric values. The value of the fuzzy game ●○ is $*$, which is less than (better for White than) any positive number, greater than any negative number, and incomparable to 0.

Formally, a game G is defined as a pair of sets of games. We represent G by listing, inside curly braces, G's left options (all the games that can result from a Black move), followed by a vertical bar and G's right options (the games that can result from a White move).[1] A game's value is itself a game in canonical form; in a certain sense, the value of G is the simplest or smallest game that is equivalent to G.

The simplest game of all, in which no moves are possible, is called 0 and is its own value. Its representation has nothing on either side of the vertical bar: $0 \equiv \{|\}$. There are more complicated zero games—that is, games that are lost by whichever player moves first—but 0 is the simplest

[1] Mnemonic: "Black" and "left" both contain the letter **l**; "white" and "right" rhyme. Recall that black is positive and white is negative.

example, and those other games are equivalent to it. Likewise, we use the symbol 1 for the simplest game conferring a one-move advantage to Black: $1 \equiv \{0 \mid \}$. Context will make clear when digit symbols stand for ordinary numbers and when for games.

Below are the values of the example games cited above, worked out in detail. (We sometimes indicate empty board positions by small dots.)

$$\begin{array}{ll} \circ\bullet\circ & = \{\circ\cdot\cdot\bullet, \bullet\cdot\cdot\circ \mid \} = \{\{|\},\{|\} \mid \} = \{\{|\} \mid \} \equiv \{0 \mid \} \equiv 1 \\ \bullet\circ\bullet\circ\bullet & = \{ \mid \circ\cdot\cdot\bullet\circ\bullet, \bullet\circ\bullet\cdot\cdot\circ\} = \{ \mid -1,-1\} = \{ \mid -1\} \equiv -2 \\ \bullet\circ & = \{\bullet \mid \circ\} = \{0 \mid 0\} \equiv * \\ \cdot & = \{ \mid \} \equiv 0 \end{array}$$

Arithmetic

We assign values to games not just to differentiate between squeaking by to victory and dealing a crushing defeat, but also so that we can compute the values of complicated games by combining the values of their noninterfering parts. In the game

$$\begin{array}{c} \cdot\bullet\circ\bullet\circ\bullet\cdot \\ \cdot\cdot\cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\circ\bullet\circ\cdot\cdot \end{array}$$

pieces from the first row can never interact with pieces from the third row. Therefore, we might as well play the first and third rows on separate boards, with each player making a move on whichever board (or whichever segment of the larger game) the player prefers. When a position can be separated into noninterfering components, the values of those components can be summed to find the value of the entire game. The value of this game is $-2 + 1 = -1$, using the values for $\bullet\circ\bullet\circ\bullet$ and $\circ\bullet\circ$ computed above.

We can *negate* a game $G = \{A, B, C, \ldots | D, E, F, \ldots\}$ by negating and reversing its components: $-G = \{-D, -E, -F, \ldots \mid -A, -B, -C, \ldots\}$. A Konane game is negated by exchanging its black and white pieces. $*$ is its own inverse: $* = -*$.

Given addition and negation, we can compare and order games. We say that

- $G > H$ iff $G - H$ ($\equiv G + -H$) is positive (is won by Black),
- $G \geq H$ iff $G - H$ is positive or zero, and
- $G = H$ iff $G \geq H$ and $H \geq G$.

The rules for assigning numeric values to games are designed so that game-theoretic summation and comparison coincide with the arithmetic versions, making numeric game values convenient to manipulate. (Explaining why this is the case is beyond the scope of this article.)

Nonintegers

Games can have noninteger numeric values. For example, Black has a half-move advantage in the game

$$\circ\bullet\circ\cdot\circ\bullet = \{0, -1 \mid 1\} = \{0 \mid 1\} \equiv \tfrac{1}{2},$$

It is a good exercise to confirm that $\circ\bullet\circ\cdot\circ\bullet + \circ\bullet\circ\cdot\circ\bullet = 1$; in other words, the second player can always win the game $\circ\bullet\circ\cdot\circ\bullet + \circ\bullet\circ\cdot\circ\bullet + \bullet\circ\bullet$, whose third component has value -1.

A game is a number iff all its options are numbers and no left option is greater than or equal to any right option.[2] The game's value is the *simplest* number strictly greater than all the left options and less than all the right options. Integers are simpler than fractions, and integers with smaller absolute values are simpler than those with larger absolute values. Among fractions, simpler ones have denominators that are smaller powers of 2. For example,

$$\{5 \mid 22\} = 6, \quad \{-22 \mid -7\} = -8, \quad \{-22 \mid 3\} = 0, \quad \{\tfrac{1}{2} \mid \tfrac{3}{4}\} = \tfrac{5}{8}, \quad \{1\tfrac{1}{4} \mid 2\} = 1\tfrac{1}{2}.$$

Why this rule works right is beyond the scope of this article.

Konane gives rise to games with arbitrarily small positive numeric values; in particular,

$$\circ\bullet \quad \bullet\circ \quad \underbrace{\quad\ \ }_{j}\!\!\bullet\circ \quad = \quad 2^{-j-1}$$

where the specified group is repeated j times, for a total of $j+5$ stones in a pattern of length $2j+7$.

Konane values may also be *infinitesimal*, with absolute value smaller than any positive number. For instance,

$$\begin{array}{ll}\bullet & \\ & \bullet\circ\end{array} = \{0 \mid *\} \equiv \uparrow$$

is a positive game, since Black wins no matter who plays first. However, Black's advantage is so slight that adding arbitrarily many of these to a game with a negative numeric value, no matter how small, still results in a negative game that White can win. Similarly, $\downarrow \equiv \{* \mid 0\} = -\uparrow$ is a negative infinitesimal.

We have only scratched the surface of the rich set of values arising in Konane and other games. These values enable us to make fine distinctions among games and are a rewarding object of study in their own right.

Simplifying Games

A game can sometimes be simplified, without changing its value, into a game with fewer options or with options that result in a shorter game. Two simplification rules—deleting dominated options and bypassing reversible moves—can substantially simplify computations with games.

A *dominated* option is worse than, or equivalent to, some other available option. No rational player will move to a dominated option, so we can remove them from consideration. Given a game $G \equiv \{A, B, C, \ldots \mid D, E, F, \ldots\}$, if $A \leq B$, then Left should move to B in preference to moving to A, and if $D \leq E$, then Right shouldn't move to E when D is available. Thus, $G = \{B, C, \ldots \mid D, F, \ldots\}$. Since options may be incomparable, simplification can leave multiple incomparable options on each side of the game.

A *reversible* move is one from which the opposing player can move to a position as least as good as the original position. For instance, given $G \equiv \{A, B, C, \ldots \mid D, E, F, \ldots\}$, if some left option of D (call it $D^L \equiv \{U, V, W, \ldots \mid X, Y, Z, \ldots\}$) is better (for Left) than G itself ($D^L \geq G$),

[2] A game in which a left option is greater than a right option is called *hot* because both players are very eager to make a move. By contrast, in a number, each left option is worse for Left than the game itself, and each right option is worse for Right than the game itself.

then Right's move to D is reversible. That is, if Right moves from G to D, then Left will certainly move to D^L (or to something even better), and Right's options will be the right options of D^L. Replacing the move from G to D by moves from G to the right followers of D^L doesn't affect the value: $G = \{A, B, C, \ldots \mid X, Y, Z, \ldots, E, F, \ldots\}$.

Proving the validity of these theorems by showing they don't change a game's value (equivalently, if G' is a simplification of G, then $G - G' = 0$) is a good exercise (see Berlekamp et al. [1982, 78]).

The simplification rules are important because an unsimplified game is just a game tree that enumerates every possible sequence of moves. Such a game tree has exponential size and provides no insight regarding the game's outcome. Combinatorial game theory

- reduces the search space by summing subgames (in general, the analysis remains exponential),
- simplifies game values into equivalent but smaller games,
- provides vocabulary for talking about game values and makes connections between apparently disparate games, and
- tells us what move is best, not just which moves win.

A Konane Rogue's Gallery

This section illustrates the principles of combinatorial game theory by analyzing some Konane positions. These positions are generally small, like those found in the endgame.

Solid Linear Patterns

Let $L(n)$ be an uninterrupted row of n Konane stones, starting with a black stone. For instance, $L(5) \equiv \bullet\circ\bullet\circ\bullet$. A pattern quickly emerges for the values of $L(n)$:

n	0	1	2	3	4	5	6	7	8	9	10
$L(n)$	0	0	$*$	-1	0	-2	$*$	-3	0	-4	$*$

We will prove by induction that, for any integer j,

$$\begin{aligned} L(2j+1) &= -j, \\ L(4j) &= 0, \\ L(4j+2) &= *. \end{aligned}$$

The table shows that these equations hold for $n \leq 10$. Every jump (which can occur only at the end of the configuration) changes $L(n)$ to $L(n-2)$, because the jumped-over stone is removed and the jumping stone is stranded away from the main game and effectively removed from play. If n is even, either player can move; but if n is odd, only White can move.

$$\begin{aligned} L(2j+1) &= \{ \mid L(2j-1)\} &&= \{ \mid -j+1\} &&= -j \\ L(4j) &= \{L(4j-2) \mid L(4j-2)\} &&= \{* \mid *\} &&= 0 \\ L(4j+2) &= \{L(4j) \mid L(4j)\} &&= \{0 \mid 0\} &&= * \end{aligned}$$

The second equality in each row follows from the inductive assumption.

Adding a single stone diagonally adjacent to a linear group makes almost no difference in the group's value:

$$
\begin{array}{llll}
LOT_1(3) \equiv & \begin{smallmatrix} & \circ & \bullet \\ \circ & & \end{smallmatrix} & = \{* \mid 0\} \equiv \downarrow & \\
LOT_1(2j) \equiv & \begin{smallmatrix} & \circ & \bullet & \circ & \bullet & \circ & \cdots \\ \circ & & & & & & \end{smallmatrix} & = j-1 & (j>0) \\
LOT_1(4j+1) \equiv & \begin{smallmatrix} & \circ & \bullet & \circ & \bullet & \cdots \\ \circ & & & & & \end{smallmatrix} & = * & (j>0) \\
LOT_1(4j+3) \equiv & \begin{smallmatrix} & \circ & \bullet & \circ & \bullet & \circ & \bullet & \cdots \\ \circ & & & & & & & \end{smallmatrix} & = 0 & (j>0)
\end{array}
$$

LOT stands for "Linear with Offset Tail"; the subscript indicates the size of the tail and the parameter n is the total number of stones in the configuration. Black can move off the left side of $LOT_1(n)$, producing two independent linear games (one with 2 stones, and one with $n-3$); or, one of the players can move off the right side of the configuration. For $j>0$, we get

$$
\begin{aligned}
LOT_1(2j) &= \{* - L(2j-3), LOT_1(2j-2) \mid \} \\
&= \{* + j - 2, LOT_1(2j-2) \mid \} \\
LOT_1(4j+1) &= \{* - L(4j-2) \mid LOT_1(4j-1)\} = \{0 \mid LOT_1(4j-1)\} \\
LOT_1(4j+3) &= \{* - L(4j) \mid LOT_1(4j+1)\} \quad = \{0 \mid LOT_1(4j+1)\}
\end{aligned}
$$

An inductive proof quickly verifies the values claimed above.

Having found values for linear positions with one-stone offset tails, we can analyze those with non-offset tails.

$$
\begin{array}{llll}
LT_1(2) \equiv & \begin{smallmatrix} \bullet \\ \circ \end{smallmatrix} & = * & \\
LT_1(3) \equiv & \begin{smallmatrix} \bullet & \circ \\ \circ & \end{smallmatrix} & = * & \\
LT_1(4) \equiv & \begin{smallmatrix} \bullet & \circ & \bullet \\ \circ & & \end{smallmatrix} & = \{* \mid \downarrow\} & \\
LT_1(4j) \equiv & \begin{smallmatrix} \bullet & \circ & \bullet & \cdots \\ \circ & & & \end{smallmatrix} & = \{* \mid -2j+2\} & (j>1) \\
LT_1(4j+1) \equiv & \begin{smallmatrix} \bullet & \circ & \bullet & \circ & \cdots \\ \circ & & & & \end{smallmatrix} & = \{2j-1 \mid *\} & (j>0) \\
LT_1(4j+2) \equiv & \begin{smallmatrix} \bullet & \circ & \bullet & \circ & \bullet & \cdots \\ \circ & & & & & \end{smallmatrix} & = \{0 \mid -2j+1\} & (j>0) \\
LT_1(4j+3) \equiv & \begin{smallmatrix} \bullet & \circ & \bullet & \circ & \bullet & \circ & \cdots \\ \circ & & & & & & \end{smallmatrix} & = \{2j \mid 0\} &
\end{array}
$$

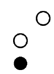
There are four possible moves from $LT_1(n)$:

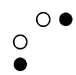
- Black can jump downward, effectively removing two stones from play and making the position linear.
- One of the players can jump over the rightmost stone.
- White can jump leftward over the leftmost black stone, effectively removing three stones from play and making the position linear.
- White can jump upward over the leftmost black stone, turning the position into $LOT_1(n-1)$.

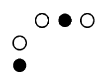
Thus, we have

$$
\begin{aligned}
LT_1(4j) &= \{-L(4j-2) \mid LT_1(4j-2), L(4j-3), LOT_1(4j-1)\} \\
&= \{* \mid LT_1(4j-2), -2j+2, 0\} \\
LT_1(4j+1) &= \{-L(4j-1), LT_1(4j-1) \mid L(4j-2), LOT_1(4j)\} \\
&= \{2j-1, LT_1(4j-1) \mid *, 2j-1\} \\
LT_1(4j+2) &= \{-L(4j) \mid LT_1(4j), L(4j-1), LOT_1(4j+1)\} \\
&= \{0 \mid LT_1(4j), 2j+1, *\} \\
LT_1(4j+3) &= \{-L(4j+1), LT_1(4j+1) \mid L(4j), LOT_1(4j+2)\} \\
&= \{2j, LT_1(4j+1) \mid 0, 2j\}
\end{aligned}
$$

A similar argument yields values for offset tails of length two, as in

$LOT_2(3) =$  $= \downarrow$

$LOT_2(4) =$  $= 0$

$LOT_2(5) =$  $= \frac{1}{2};$

and, for $j > 0$,

$$
\begin{aligned}
LOT_2(4j) &= \{L(4j-1), -1+L(4j-4) \mid -L(4j-2), LOT_2(4j-2)\} \\
&= \{-2j+1, -1+0 \mid LOT_2(4j-2), *\} \\
&= \{-1 \mid -1\} = -1 + * \\
LOT_2(4j+1) &= \{L(4j), -1+L(4j-3), LOT_2(4j-1) \mid -L(4j-1)\} \\
&= \{0, -1-2j+2, LOT_2(4j-1) \mid 2j-1\} \\
&= \{2j-3 \mid 2j-1\} = 2j-2 \\
LOT_2(4j+2) &= \{L(4j+1), -1+L(4j-2) \mid LOT_2(4j), -L(4j)\} \\
&= \{-2j, -1+* \mid LOT_2(4j), 0\} \\
&= \{-1+* \mid -1+*\} = -1 \\
LOT_2(4j+3) &= \{L(4j+2), -1+L(4j-1), LOT_2(4j+1) \mid -L(4j+1)\} \\
&= \{*, -1-2j+1, LOT_2(4j+1) \mid 2j\} \\
&= \{2j-2 \mid 2j\} = 2j-1
\end{aligned}
$$

A similar but more complicated analysis yields results for LT_2 and LT_3:

$$
\begin{array}{lcrlcr}
LT_2(2j+1) & = & -j & \qquad LT_3(2j) & = & -j+3 \\
LT_2(4j) & = & 0 & \qquad LT_3(4j+1) & = & 2+* \\
LT_2(4j+2) & = & * & \qquad LT_3(4j+3) & = & 2
\end{array}
$$

Other Values Generated by Konane

Konane gives rise to some other interesting game-theoretic values, such as

$$\bigcirc\quad\bigcirc\bullet\quad\bullet\quad\bullet\bigcirc\quad\bigcirc\quad\bigcirc\bullet\quad\bullet \quad = \quad *2$$

The values 0, $* \equiv \{0 \mid 0\}$, $*2 \equiv \{0, * \mid 0, *\}$, $*3 \equiv \{0, *, *2 \mid 0, *, *2\}$, ... are called *nimbers* because they are the values that arise in the game of Nim.[3] Nimbers are fuzzy and incomparable to one another, and when they are added, powers of 2 cancel:

$$* + * = 0, \qquad * + *2 = *3, \qquad *6 + *3 = *5.$$

Nimbers above $*$ rarely arise in *partizan* games in which the options of the two players differ; so it is interesting that it only takes four stones to produce the value $*2$ (see $D2(4)$ below). We have not yet discovered Konane positions worth $*4$.

The following values hold only for Modern Konane, which prohibits multiple jumps in a move.

$$
\begin{array}{rcl}
\bigcirc\bullet\cdot\cdot\underbrace{\bigcirc\bullet}_{j}\bigcirc\cdot\bigcirc & = & \{j \mid 0\} \\
\bigcirc\bullet\cdot\bullet\cdot\cdot\underbrace{\bigcirc\bullet}_{j}\bigcirc\cdot\bigcirc & = & \{j \mid \{j \mid 0\}\} = j + +_j
\end{array}
$$

The values $+_x \equiv \{0 \mid \{0 \mid -x\}\}$ (pronounced "tiny x") are a family of positive infinitesimals that, for positive noninfinitesimal x, are infinitely smaller than $\uparrow \equiv \{0 \mid *\}$. (You can verify this by showing that $+_x + +_x + \cdots + +_x + \downarrow < 0$.) Its inverse is $-_x \equiv \{\{x \mid 0\} \mid 0\} = -+_x$ (pronounced "miny x").

While it isn't trivial to convert these into Ancient Konane positions with the same value, we can demonstrate other positions with tiny values under both variations of the rules. For instance,

$$
\begin{array}{cccc}
\bullet & \bigcirc & \cdot & \cdot \\
\cdot & \cdot & \bigcirc & \cdot \\
\cdot & \bigcirc & \cdot & \bigcirc
\end{array}
\;=\; +_1, \qquad
\begin{array}{ccccc}
\bullet & \bigcirc & \cdot & \cdot & \cdot \\
\cdot & \cdot & \bigcirc & \cdot & \bigcirc \\
\cdot & \bigcirc & \cdot & \bigcirc & \cdot \\
\cdot & \cdot & \cdot & \cdot & \bigcirc
\end{array}
\;=\; +_2, \qquad \text{etc.}
$$

Double Rows

Two rows of stones placed side by side are considerably more difficult to generalize, because each move can affect many other moves.

[3] Nim [Bouton 1901] is played with heaps of game pieces. Each move consists of removing any number of pieces from one heap, possibly eliminating that heap. The first player unable to move loses (equivalently, the player who removes the last game piece wins).

$$\underbrace{\begin{array}{|lllll}\hline \bullet & \circ & \bullet & \circ & \bullet \\ \circ & \bullet & \circ & \bullet & \\ \bullet & \circ & \bullet & & \\ \circ & \bullet & & & \\ \bullet & & & & \end{array}}_{n} = \begin{cases} \frac{n}{2}\left(\frac{n}{2}+1\right) - 1, & n \text{ even;} \\ -\left(\frac{n+1}{2}\right)^2 + 2, & n \text{ odd, } n > 1. \end{cases}$$

Figure 1: The values of symmetric triangular corner positions.

For instance, $L2(n)$ is a double linear row of stones:

$$L2(12) = \begin{array}{cccccc} \bullet & \circ & \bullet & \circ & \bullet & \circ \\ \circ & \bullet & \circ & \bullet & \circ & \bullet \end{array} = \begin{cases} *2 & \text{ancient rules} \\ 0 & \text{modern rules} \end{cases}$$

The first few values of $L2(n)$ are

n	0	2	4	6	8	10	12	14	16
$L2(n)$	0	$*$	$*$	0	0	$*$	$*2$ or 0	0	0

$D2(n)$ is a double diagonal row of stones, such as

$$D2(6) = \begin{array}{cccc} \bullet & \circ & & \\ & \bullet & \circ & \\ & & \bullet & \circ \end{array} = \begin{cases} *3 & \text{ancient rules} \\ 0 & \text{modern rules,} \end{cases}$$

giving rise to the following values under ancient and modern rules:

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
ancient	$*$	$*2$	0	$*3$	$*$	$*$	0	0	$\{0, * \mid 0, \{0, * \mid 0, \uparrow + *\}\}$	$*$	$*2$	0	$\{0 \mid *, *2\}$	$*$	$*$
modern	$*$	$*2$	0	0	$*$	$*$	$\uparrow$	0	$\{0, * \mid 0, \{0, * \mid 0, \uparrow + *\}\}$	$*$	$*$	0	0	$*2$	$*$

Patterns in Two Dimensions

In real Konane games, play frequently proceeds to the corners after the center of the board has been largely cleared. It is a bad idea to pen one's opponent into a corner of the board; when a number of diagonals are filled, the player whose stones are not on the outermost row gets a point for every jumpable enemy stone save one (see Figure 1). The winner cannot jump the corner stone (if it belongs to the enemy), and one other enemy stone on the second or third diagonal from the corner also is not jumped.

How to Separate, or Peg Solitaire Penetration

A key feature of combinatorial game theory is that a game's value is the sum of the values of its noninterfering parts. This observation can substantially simplify game-theoretic analysis.

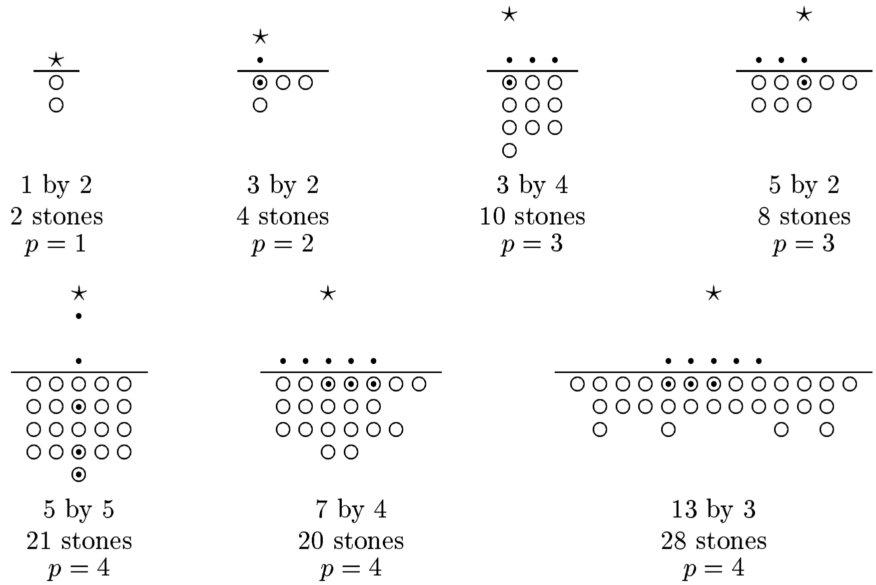


Figure 2: Minimal configurations for achieving penetrations. The circles represent original stone positions, and penetration occurs upward across the horizontal line to the star. The small dots represent one possible intermediate stone position—usually a previously shown configuration, shifted upward by one space.

Because pieces move on the game board, it can be difficult to separate a Konane game into independent subparts. Games such as Domineering [Berlekamp et al. 1982, 117–120] are monotonic: pieces are only removed, never moved or added, so each connected component makes a separate, independent game. In Konane, nonconnected configurations separated by empty rows or columns can interact. In the game $\bullet \circ \cdot \cdot \bullet \circ$, each of the two-stone groups has value $*$; but under modern rules, the entire configuration has value $\{*, \uparrow \mid *, \downarrow\}$ rather than 0.

The greatest distance away from a group of stones at which one of its members may end up is its *penetration*. More precisely, if we place the group on one side of a line, the group's penetration is the greatest distance beyond the line at which some stone can end up after an arbitrary sequence of Konane jumps (not necessarily an alternating sequence of moves by White and Black). A 1-by-2 group can penetrate a single space in its lengthwise direction. Thus, if two such groups in the proper relative orientation are separated by two spaces, as in $\bullet \circ \cdot \cdot \bullet \circ$, they can interact. If two groups are separated by more space than the sum of their penetrations, they cannot interact (unless one or both interacts with some other group).

Given this rule for separating groups, we need a way to determine their penetrations. Tighter bounds on potential penetration will permit more frequent separation of positions into independent parts.

Positive Results (Lower Bounds)

Figure 2 shows how to construct stone configurations with penetrations of up to 4 spaces. The horizontal line is the boundary across which penetration must occur.

Table 1 gives (identical) upper and lower bounds for the penetrations achievable by stone configurations of various sizes. The entries for (3,4) and (4,3) indicate that any configuration of

Table 1: Maximum penetration of rectangular stone or peg configurations.

	Width												
Depth	1	2	3	4	5	6	7	8	9	10	11	12	13+
1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	2	2	3	3	3	3	3	3	3	3	3
3	1	1	2	2	3	3	3	3	3	3	3	3	4
4	1	1	3	3	3	3	4	4	4	4	4	4	4
5+	1	1	3	3	4	4	4	4	4	4	4	4	4

stones that fits entirely within a 3-by-4 rectangle can penetrate no more than 3 spaces parallel to its long side and no more than 2 spaces parallel to its short side—and there exist configurations that achieve these penetrations. Similarly, no configuration of depth 2, regardless of its width, can achieve penetration greater than 3. These results fill in the gaps in previous expositions that approached the problem from the point of view of a peg solitaire army attempting to send a scout into a desert [Berlekamp et al. 1982, 715].

Figure 2 shows a sample of the achievable configurations (in each case, the narrowest possible configuration, given a particular depth). Since adding more stones cannot reduce a configuration's penetration, any element in Table 1 can be constructed by adding dummy rows to one of the configurations shown in Figure 2. A similar, but reversed, argument is used in the next subsection (removing stones cannot increase a group's penetration), so that, again, only a small number of results need be proven.

Table 1's lower bounds are pessimistic for two separate reasons. First, it does not account for the structure of a stone group, only its bounding rectangle. A 3-by-4 configuration needs at least 10 stones to achieve penetration 3, and not all 3-by-4 configurations containing 11 stones achieve that penetration. Figure 2 gives lower bounds on the number of stones required to achieve a penetration, which can be used in concert with Table 1 to improve its bounds.

The second source of imprecision is that comparing group separations with sums of penetrations reveals which groups cannot possibly interact via any line of play, but not which groups might interact given rational play. High penetrations rarely occur in Konane games, since it is unlikely that the moves that are best for each player coincide with the moves required for maximal penetration. Since so much cooperation between the players is required, achieving maximal penetration is more like a puzzle or a cooperative game than a competitive one. (This is why Figure 2 depicts all stones.)

As an example, consider the game:

○ ● · · · · ○ ●
● ○ · · · · ● ○
○ ● · · · · ○ ●

Each of the stone groups has penetration 2 toward the other, so there is a line of play that causes the two components to interact and we cannot consider the components (each of which has value 0) separately. However, in this game it is foolish to move next to an opposing piece that has achieved penetration 2, so no sane player will cause the two components to interact;

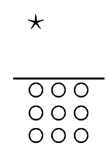
the subgames are effectively independent.[4] A set of correct (if occasionally pessimistic) rules that can be automated or applied mindlessly is preferable to such case-by-case reasoning, however. Determining separability is complicated and time-consuming, but worthwhile: it substantially reduces the overall work required in a computer program [Wolfe 1994] that evaluates Konane positions.

Negative Results (Upper Bounds)

No greater penetration than that claimed in Table 1 is possible. The proof relies on a *potential function* (sometimes called a *pagoda function* [Beasley 1962, 1985; Berlekamp et al. 1982, 712], which assigns potentials to stones depending on their location. The potentials are chosen so that no jump increases the total potential of the stones on the board. That is, after a jump (which removes stones from two locations and adds a stone to a third location), the net sum of the potentials of all stones in a configuration is less than or equal to its pre-jump potential.

Since the total potential of a configuration of stones is monotonically nonincreasing, if that total is less than the potential of a particular location, then no sequence of jumps from the configuration can result in a stone at the location.

For instance, to prove that we can't get from a 3-by-3 configuration of stones to the star at penetration 3 in



we use the potential function

21	13	8
13	8	5
8	5	3
5	3	2
3	2	1
2	1	1

The initial potential is 20 and the desired potential is 21. Since no jump increases the potential, the goal is impossible.

In most of the figures that follow, the goal is assigned a potential of 1 and we show that the configuration's initial potential is less than 1.

The most interesting negative result is:

No matter how many pieces are provided, it is impossible to penetrate 5 spaces.

To show this, we start with an infinite two-dimensional configuration of stones (see Figure 3) and show that any fewer stones (in particular, any finite number) is not enough. (In this and subsequent figures, the vertical boundary line is to be crossed from the left.) The figure is understood to extend infinitely in each direction; every location to the left of the line contains a stone.

[4] In this example, a symmetry argument also shows that the entire position is worth zero. The second player's winning strategy (called *Tweedledum–Tweedledee* [Berlekamp et al. 1982, 5]) is to respond to each move by rotating the board 180° and then mimicking the opponent's move.

$$
\begin{array}{ccccc|ccccc}
 & & \sigma^{13} & \sigma^{12} & \sigma^{11} & & & & & \\
 & \sigma^{13} & \sigma^{12} & \sigma^{11} & \sigma^{10} & & & & & \\
\sigma^{13} & \sigma^{12} & \sigma^{11} & \sigma^{10} & \sigma^{9} & & & & & \\
\sigma^{12} & \sigma^{11} & \sigma^{10} & \sigma^{9} & \sigma^{8} & \sigma^{7} & & & & \\
\sigma^{11} & \sigma^{10} & \sigma^{9} & \sigma^{8} & \sigma^{7} & \sigma^{6} & \sigma^{5} & & & \\
\sigma^{10} & \sigma^{9} & \sigma^{8} & \sigma^{7} & \sigma^{6} & \sigma^{5} & \sigma^{4} & \sigma^{3} & \sigma^{2} & \\
\sigma^{9} & \sigma^{8} & \sigma^{7} & \sigma^{6} & \sigma^{5} & \sigma^{4} & \sigma^{3} & \sigma^{2} & \sigma^{1} & 1 \\
\sigma^{10} & \sigma^{9} & \sigma^{8} & \sigma^{7} & \sigma^{6} & \sigma^{5} & \sigma^{4} & \sigma^{3} & \sigma^{2} & \\
\sigma^{11} & \sigma^{10} & \sigma^{9} & \sigma^{8} & \sigma^{7} & \sigma^{6} & \sigma^{5} & & & \\
\sigma^{12} & \sigma^{11} & \sigma^{10} & \sigma^{9} & \sigma^{8} & \sigma^{7} & & & & \\
\sigma^{13} & \sigma^{12} & \sigma^{11} & \sigma^{10} & \sigma^{9} & & & & & \\
 & \sigma^{13} & \sigma^{12} & \sigma^{11} & \sigma^{10} & & & & & \\
 & & \sigma^{13} & \sigma^{12} & \sigma^{11} & & & & &
\end{array}
$$

Figure 3: A finite army of peg solitaire pieces cannot achieve a penetration of 5.

The value σ is the inverse of the golden ratio, or $(\sqrt{5}-1)/2 \doteq .618034$. This satisfies

$$\sigma^2 + \sigma \quad = \quad 1$$

and, more generally,

$$\sigma^{n+2} + \sigma^{n+1} \quad = \quad \sigma^n \ . \tag{1}$$

This is a particularly good potential function because no other function decreases values so quickly and uniformly while satisfying the summation constraint.

Summing the geometric series or induction and (1) reveal that

$$\sum_{i=n}^{\infty} \sigma^i \quad = \quad \sigma^{n-2} \ , \tag{2}$$

so the rows partly shown to the left of the vertical line in Figure 3 have total potentials σ^9, σ^8, $\ldots$, σ^4, σ^3, σ^4, $\ldots$, σ^8, and σ^9.

The sequence of row totals can itself be manipulated by (1) and (2), yielding

$$\sum_{i=3}^{\infty} \sigma^i + \sum_{i=4}^{\infty} \sigma^i = \sigma + \sigma^2 = 1.$$

Thus, it is theoretically possible to place a stone in the fifth column, given a half-plane of stones; however, if even one of the stones is missing, the potentials sum to less than 1 and the goal is impossible.

Showing that no configuration of width 4 can achieve a penetration of 4 is similar (see Figure 4). The same potential function is used; the boxed potentials sum to 1, demonstrating that a location 4 spaces out along the top center row is inaccessible. Shifting the box down (so the goal is a location along the top row, or is in a row above that) can only decrease the original potential, so none of the locations 4 spaces to the right of the configuration and at or above the 1 is accessible. By symmetry, the locations four spaces out along or below the bottom center row are inaccessible; so no location 4 spaces distant can be reached.

$$
\begin{array}{cccccc|cccc}
 & \sigma^{10} & \sigma^{9} & \sigma^{8} & \sigma^{7} & \sigma^{6} & \sigma^{5} & & & \\
 & \sigma^{9} & \sigma^{8} & \sigma^{7} & \sigma^{6} & \sigma^{5} & \sigma^{4} & \sigma^{3} & \sigma^{2} & \\
\cdots & \sigma^{8} & \sigma^{7} & \sigma^{6} & \sigma^{5} & \sigma^{4} & \sigma^{3} & \sigma^{2} & \sigma^{1} & 1 \\
 & \sigma^{9} & \sigma^{8} & \sigma^{7} & \sigma^{6} & \sigma^{5} & \sigma^{4} & \sigma^{3} & \sigma^{2} & \\
 & \sigma^{10} & \sigma^{9} & \sigma^{8} & \sigma^{7} & \sigma^{6} & \sigma^{5} & & & \\
 & \sigma^{11} & \sigma^{10} & \sigma^{9} & \sigma^{8} & \sigma^{7} & & & &
\end{array}
$$

Figure 4: No configuration of width 4 can achieve a penetration of 4.

The same argument shows that the penetration of a configuration with depth 2 is no more than 3 (see Figure 5).

$$
\begin{array}{cc|cccc}
\vdots & & & & & \\
\sigma^{8} & \sigma^{7} & & & & \\
\sigma^{7} & \sigma^{6} & \sigma^{5} & & & \\
\sigma^{6} & \sigma^{5} & \sigma^{4} & \sigma^{3} & \sigma^{2} & \\
\sigma^{5} & \sigma^{4} & \sigma^{3} & \sigma^{2} & \sigma^{1} & 1 \\
\sigma^{6} & \sigma^{5} & \sigma^{4} & \sigma^{3} & \sigma^{2} & \\
\sigma^{7} & \sigma^{6} & \sigma^{5} & & & \\
\sigma^{8} & \sigma^{7} & & & & \\
\vdots & & & & &
\end{array}
$$

Figure 5: No configuration of depth 2 can achieve a penetration of 4.

A different potential function, shown in Figure 6, can be useful with small configurations. In the three cases shown in the figure, by applying translation and symmetry, every location in the rightmost column is inaccessible.

The 7-by-3 case (see Figures 7–8) requires two different potential functions to show that all locations in the fourth column are inaccessible. The middle location (as well as a few others) is off limits because of the potential function of Figure 7. The box shown is symmetric and cannot be shifted without its potential exceeding 1. The original potential function, while it cannot show that the center location is inaccessible, can show that all other positions in the fourth column are inaccessible (Figure 8).

6 by 4:

$$
\begin{array}{cccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma^7 & \sigma^6 & \sigma^5 & \sigma^4 & \sigma^3 & \sigma^2 & \sigma^1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma^7 & \sigma^6 & \sigma^5 & \sigma^4 & \sigma^3 & \sigma^2 & \sigma^1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma^7 & \sigma^6 & \sigma^5 & \sigma^4 & \sigma^3 & \sigma^2 & \sigma^1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma^7 & \sigma^6 & \sigma^5 & \sigma^4 & \sigma^3 & \sigma^2 & \sigma^1 & 1
\end{array}
$$

4 by 3:

$$
\begin{array}{ccc|ccc}
0 & 0 & 0 & 0 & 0 & 0 \\
\sigma^5 & \sigma^4 & \sigma^3 & \sigma^2 & \sigma^1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\sigma^5 & \sigma^4 & \sigma^3 & \sigma^2 & \sigma^1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\sigma^5 & \sigma^4 & \sigma^3 & \sigma^2 & \sigma^1 & 1
\end{array}
$$

2 by n:

$$
\begin{array}{cccc|cc}
 & 0 & 0 & 0 & 0 & 0 \\
\cdots & \sigma^4 & \sigma^3 & \sigma^2 & \sigma^1 & 1 \\
 & 0 & 0 & 0 & 0 & 0 \\
 & \sigma^4 & \sigma^3 & \sigma^2 & \sigma^1 & 1
\end{array}
$$

Figure 6: Other small configurations; the group cannot penetrate to the rightmost column.

$$
\begin{array}{|ccc||cccc}
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma^6 & \sigma^5 & \sigma^4 & \sigma^3 & \sigma^2 & \sigma^1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma^6 & \sigma^5 & \sigma^4 & \sigma^3 & \sigma^2 & \sigma^1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma^6 & \sigma^5 & \sigma^4 & \sigma^3 & \sigma^2 & \sigma^1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
$$

Figure 7: The 7-by-3 case: This potential function shows that the center position in the fourth column is inaccessible.

$$
\begin{array}{|ccc||cccc}
\hline
\sigma^{10} & \sigma^9 & \sigma^8 & & & & \\
\sigma^9 & \sigma^8 & \sigma^7 & \sigma^6 & & & \\
\sigma^8 & \sigma^7 & \sigma^6 & \sigma^5 & \sigma^4 & & \\
\sigma^7 & \sigma^6 & \sigma^5 & \sigma^4 & \sigma^3 & \sigma^2 & \\
\sigma^6 & \sigma^5 & \sigma^4 & \sigma^3 & \sigma^2 & \sigma^1 & 1 \\
\sigma^7 & \sigma^6 & \sigma^5 & \sigma^4 & \sigma^3 & \sigma^2 & \\
\sigma^8 & \sigma^7 & \sigma^6 & \sigma^5 & & & \\
\hline
\end{array}
$$

Figure 8: The 7-by-3 case: A different potential function shows that all locations in the fourth column are inaccessible, except for the center one.

A different, less uniform potential function suffices to show that 11-by-3 (and smaller) configurations can't penetrate four spaces; see Figure 9.

The potential function of Figure 9 also shows that, for a 12-by-3 stone configuration, no more than the two center stones of the fourth column are accessible. The maximum potential initial potential of a 12-by-3 group is 8—which is achieved only for the two 12-by-3 groups that entirely contain the 11-by-3 group shown in Figure 9. The potential function of Figure 10 establishes that the remaining two locations are off limits.

Previous Technical Papers

Konane has received little attention in the technical literature. Gyllenskog [1976] used the game in teaching artificial intelligence search techniques; most of the student-written programs routinely defeated human novices. He states that the branching factor in the middle game is approximately 10 and rarely exceeds 10 or 12. He calls his static evaluation function "very good"

$$
\begin{array}{ccc|cccc}
0 & \sigma^6 & \sigma^6 & 2\sigma^6 & 3\sigma^6 & 5\sigma^6 & \\
0 & \sigma^5 & \sigma^5 & 2\sigma^5 & 3\sigma^5 & 5\sigma^5 & \\
0 & \sigma^4 & \sigma^4 & 2\sigma^4 & 3\sigma^4 & 5\sigma^4 & \\
0 & \sigma^3 & \sigma^3 & 2\sigma^3 & 3\sigma^3 & 5\sigma^3 & \\
0 & \sigma^2 & \sigma^2 & 2\sigma^2 & 3\sigma^2 & 5\sigma^2 & \\
0 & \sigma^1 & \sigma^1 & 2\sigma^1 & 3\sigma^1 & 5\sigma^1 & \\
0 & 1 & 1 & 2 & 3 & 5 & 8 \\
0 & \sigma^1 & \sigma^1 & 2\sigma^1 & 3\sigma^1 & 5\sigma^1 & \\
0 & \sigma^2 & \sigma^2 & 2\sigma^2 & 3\sigma^2 & 5\sigma^2 & \\
0 & \sigma^3 & \sigma^3 & 2\sigma^3 & 3\sigma^3 & 5\sigma^3 & \\
0 & \sigma^4 & \sigma^4 & 2\sigma^4 & 3\sigma^4 & 5\sigma^4 & \\
0 & \sigma^5 & \sigma^5 & 2\sigma^5 & 3\sigma^5 & 5\sigma^5 & \\
0 & \sigma^6 & \sigma^6 & 2\sigma^6 & 3\sigma^6 & 5\sigma^6 &
\end{array}
$$

Figure 9: The 11-by-3 case: No location in the fourth row is accessible. For 12-by-3 configurations, this potential function shows that all but two locations in the fourth column inaccessible.

$$
\begin{array}{ccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 1 & 1 & 2 & 3 & 5 & \\
0 & 1 & 1 & 2 & 3 & 5 & \\
0 & 2 & 2 & 4 & 6 & 10 & \\
0 & 3 & 3 & 6 & 9 & 15 & \\
0 & 5 & 5 & 10 & 15 & 25 & \\
0 & 8 & 8 & 16 & 24 & 40 & 64 \\
0 & \frac{24}{5} & \frac{24}{5} & \frac{48}{5} & \frac{72}{5} & 24 & \\
0 & \frac{16}{5} & \frac{16}{5} & \frac{32}{5} & \frac{48}{5} & 16 & \\
0 & \frac{8}{5} & \frac{8}{5} & \frac{16}{5} & \frac{24}{5} & 8 & \\
0 & \frac{8}{5} & \frac{8}{5} & \frac{16}{5} & \frac{24}{5} & 8 & \\
0 & 0 & 0 & 0 & 0 & 0 &
\end{array}
$$

Figure 10: The center two locations four spaces distant from a 12-by-3 stone configuration are inaccessible.

but does not describe it. (One obvious, but flawed, static evaluation function simply compares the number of stones of each color remaining on the board.) Gyllenskog gives a sample final position (Figure 1 in his paper), which we can analyze game-theoretically to assign a value of 5 (4, under modern rules).

De Oliveira [1981] pursues Gyllenskog's suggestion of using Konane to explore search techniques. The complete games given in her appendices permit us to evaluate her program by analyzing some moves game-theoretically.[5] For instance, her p. C.7 shows a board before and after a Black move (the only move so displayed). This particular move is not very good, as it throws away most of Black's advantage: The position before the move is worth 3 (a win for Black), while the position after the move is worth 0 (a razor-thin win for the second player to move—which is Black, since White is now obliged to respond to Black's move).

Future Work

Much work remains to be done on Konane. A dictionary of Konane positions would permit quick analysis and insight into the structure of the game. The author and others have begun this work, but many more configurations remain to be analyzed. Almost no work has been done on game positions that include the edge of the board.

Under what conditions do Modern and Ancient Konane produce different results? Which of the games is really simpler? Modern Konane restricts the permissible moves, but sometimes a position's ancient value is simpler than its modern value. An automatic technique for converting Modern Konane positions into Ancient Konane positions with the same value, and vice versa, would be helpful.

The rules for separation of Konane games into sums of smaller games can be extended. What are the maximal penetrations that occur in best play, and when can theoretically inseparable groupings actually be considered independently, because real play won't result in their interaction? Also, why is a position's width more important than its depth in determining its penetration?

It would be interesting to find global strategies for Konane, which could be useful at points where a game-theoretic valuation of the position is too complicated to be feasible. For instance, is it good, bad, or indifferent to strand one's own (or one's opponent's) pieces away from the action? Another line of investigation is solving Konane for small game boards. For instance, the second player always wins games played on a 4-by-4, 4-by-5, or 4-by-6 board; but a game on a 5-by-5 board is won by the player whose pieces are in the corners of the board.

The author has extended the computer program Gamesman's Toolkit [Wolfe 1994], which performs game-theoretic operations, to analyze both Ancient Konane and Modern Konane positions. This program eliminates the tedious and error-prone task of computing and simplifying game values.

Finally, there is a world of other games to study. Berlekamp et al. [1982] is a good source for examples; Wolfe [1995] also suggests some directions for study. Or choose your favorite game and see where it leads you!

[5] Since I do not speak Portuguese, I was unable to determine whether de Oliveira detailed a static evaluation function.

Appendix: Anthropological Details

Konane was played in preliterate Hawaii [Sizer 1991]. Konane was popular among all classes, and men and women often played together, unlike some other Hawaiian games that were *tapu* (taboo) among the common people or were played by only one sex [Emory 1924]. Konane was a particular favorite of old men [Buck 1957, 317]. A game sometimes lasted an entire day; in a match, often a large number of games were played before determining the winner [Emory 1924].

Captain James Cook, who in 1778 during his third voyage was the first European to visit Hawaii, described a native game that is clearly Konane:

> One of their games resembles our game of draughts [checkers]; but, from the number of squares, it seems to be much more intricate. The board is of the length of about two feet, and is divided into two hundred and thirty-eight squares, fourteen in a row [hence a 14-by-17 board]. In this game they use black and white pebbles, which they move from one square to another. [Cook and King 1784, 312].

Like other indigenous Hawaiian games and sports, Konane declined in popularity after the arrival of Westerners; by 1924, only one ninety-year-old Hawaiian native woman was known to be acquainted with the game [Emory 1924]. In the last few decades, however, native pastimes have begun a resurgence. Today, students in schools emphasizing Hawaiian culture learn to play Konane as early as first or second grade [Kawai`ae`a 1995]. The loss of popularity resulted in part from Hawaiians' enthusiastic acceptance of novel foreign games, but was primarily due to the efforts of Christian missionaries [Mitchell 1982, 180]. The missionaries taught that Hawaiian culture and customs were inferior, eliminated religious practices that permeated games and sports, and criticized adults who engaged in play during daytime "working" hours. Missionaries also strove to stamp out games associated with gambling. Hawaiians were "greatly addicted to gambling" [Cook and King 1784], and the betting in Konane was sometimes very heavy [Emory 1924], a practice Brigham [1892, 54] says began in the second half of the 18th century. Alexander [1871, 88] calls gambling the chief purpose of Hawaiian games.

Konane was played on a rectangular grid of indentations or holes. (Corney [1896] describes Hawaiian "draughts"—probably Konane—boards as painted in squares; Andrews [1865] says Konane was played on squares of black and white.) Pitted slabs with rows of holes for playing Konane were found on the front of the platform of many houses [Emory 1924], and portable game boards were common [Buck 1957]; the game was also frequently played on the plaits of the lauhala mat [Emory 1924]. The twenty Konane slabs and boards reported by Emory [1924] and Buck [1957] average 134 holes each, with a geometric mean of 125 holes. The number of rows ranged from 8 to 13, and the number of columns from 8 to 20; five of the boards were square and the remainder rectangular. The board (*papa konane*) is set end on between the players, with the longer dimension between them.

The center of the board was called *piko* (navel) and frequently marked with an inset human molar; sometimes every position had an inset tooth (or a chicken or human bone [Brigham 1892, 60]). The row along the borders of the board was termed *kaka`i*. Before starting play, the board positions were filled with alternating black and white stones. Local beaches provided basalt and coral pebbles for game pieces, whose preferred size was under an inch in diameter and slightly flattened rather than spherical. The names for the black and white stones are variously reported as *`ili `ele`ele* (black-skinned) and *`ili kea* (white-skinned) [Buck 1957], as *`ili`ili `ele`ele* and

ʻiliʻili keʻokeʻo [Mitchell 1975], as *ka eleele* (or *ele*) and *ke keokeo* (or *kea*) [Emory 1924], and as *hiu* [Andrews 1865].

To begin the game, the players decide who picks up the first stone, which must be the center stone, one laterally next to it, or one at a corner. The second player picks up an adjacent stone of the other color; if the first player selected a stone adjacent to the center, then the second player must take the center stone. Thereafter, the players take turns jumping with the color that each initially picked up, removing jumped-over stones of the other color (*lawe ili keokeo, paani, ka eleele*: "removing the whites is playing with the blacks"). The first player unable to move loses (*make*); the other player wins (*ai*).

A move consists of jumping (*holo* or *konene*) one's own piece over an adjacent enemy piece into an empty location just beyond; the enemy piece is removed. Jumping occurs along a row or file, never diagonally and never in two directions on a single move. Multiple enemy pieces may be removed providing they are all on the same row or file, they are separated by one empty location, and there is a vacant position at the end of the line; such a move is called *kaholo*. *Kuʻi* (strike back) means to jump over the piece just moved by the opponent, along the same row or file but in the opposite direction.

An alternative name for Konane was *mu*, and for the board, *papamu*. Brigham [1906] notes that *mu* was the name of the official who captured men for sacrifice or for judicial punishment and suggests this name was adopted for the game. Buck [1957] thinks it more likely that this *mu*, and *papamu*, come from the English word "move," which Europeans frequently said as they played board games. (The New Zealand Maoris used the name *mu* for checkers for just this reason.)

Andrews [1865] says *pahiuhiu* was the name of a game like Konane, which is a species of *punipeke*. A more authoritative Hawaiian dictionary [Pukui and Elbert 1957] states that *punipeki* is a game similar to Fox and Geese whose game pieces are moved on the board by pushing them with sharp sticks; *pahiuhiu* (or *pahuihui*) is a game of throwing darts at a target, or, as a verb, to push a stone with sharp sticks toward a goal. The spelling "*pahuuhiu*" in Murray [1952] appears to be a typographical error.

The ancient Javanese/Malayan game of *main chuki* or *tjuki* is similar to Konane in that it is a kind of checkers played with 60 white beans and 60 black beans on the 120 points formed by intersections of lines [Wilken 1893, 162; Wilkinson 1925, 60]. The English game of Leap Frog is also similar, except that during a move a piece may make any number of jumps, even in orthogonal directions, and the winner is the player who captures the greatest number of playing pieces rather than the one whose opponent is blocked from further moves [McConville 1974].

Acknowledgments

Elwyn Berlekamp introduced me to the beautiful theory of combinatorial games, encouraged me to pursue research in this area, and provided valuable comments and insights on this work. Paul Campbell provided patient support, helpful suggestions, and assisted with references, including locating the exact reference to Konane in perhaps the only one of the many editions of Cook's writings that contains it. Darrah P. Chavey made helpful suggestions on the manuscript and found the potential functions showing that 8-by-3 through 12-by-3 stone configurations cannot penetrate four spaces. (Chavey's potential functions are more persuasive than the author's exhaustive computer search that had previously demonstrated those results.) Ellen Spertus's comments also improved this article. The bulk of this work was done while the author was at MIT;

other parts were done at Microsoft Research.

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